Nonlinear Anisotropic Diffusion in Positron Emission Tomography Kinetic Imaging

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Introduction

This study considers post-processing of global kinetic images derived from data collected with Positron Emission Tomography (PET). Global kinetic images are not acquired images, but rather represent the solution to an inverse problem solved pixel-by-pixel. Specifically, kinetic images are solutions to the following inverse problem,

$$y_j(t;k_j) = f(t;k_j) * u(t), \tag{1}$$

where k_j is a vector of parameters to be determined for each pixel j, $f(t;k_j)$ is an analytic function with known form, nonlinear in k_j, $y_j(t;k_j)$ is the pixel concentration function, and $u(t)$ is a pixel-invariant input function.

If we let $\mathbf{v}$ denote the original true global kinetic image, the typical formulation for an image restoration problem is to consider the relation

$$\mathbf{v_0} = R\mathbf{v} + \eta$$

where $\mathbf{v_0}$ is the observed image, R is a blur operator, and η is the additive noise. Usually, the blur operator R describes the deterministic process of image acquisition. The noise is also characterized by the signal transmission. Although this is not strictly true of kinetic imaging, this study performs image restoration for kinetic imaging under these general assumptions.

Anisotropic Diffusion

Nonlinear anisotropic diffusion in the context of image restoration is traced back to the work of Perona and Malik [1], who refer to the following model as the anisotropic diffusion (AD) model,

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= div(c(|\nabla \mathbf{v}|)\nabla \mathbf{v}), \text{ in } \Omega \times (0,T), \\ \frac{\partial \mathbf{v}}{\partial N} &= 0, \text{ on } \partial\Omega \times (0,T), \\ \mathbf{v}(x,y,0) &= \mathbf{v_0}(x,y) \text{ in } \Omega, \end{aligned} \tag{2}$$

where N is the outward normal to the boundary $\partial\Omega$ and $c(\cdot)$ is a nonnegative, smooth decreasing function of the magnitude of the local image gradient $|\nabla \mathbf{v}| = \sqrt{\mathbf{v_x^2} + \mathbf{v_y^2}}$. For constant functions $c(\cdot) = cst$, this reduces to the isotropic heat equation. In the context of image restoration, t acts as an iteration number. The discretization adopted in [1] and used in this study is a forward Euler method, using a box nearest-neighbor spatial discretization,

$$\begin{aligned} v_{i,j}^{t+1} = v_{i,j}^t + \lambda[&c_{i+\frac{1}{2},j}(v_{i+1,j}^t - v_{i,j}^t) + c_{i,j+\frac{1}{2}}(v_{i,j+1}^t - v_{i,j}^t) \\ &+ c_{i-\frac{1}{2},j}(v_{i-1,j}^t - v_{i,j}^t) + c_{i,j-\frac{1}{2}}(v_{i,j-1}^t - v_{i,j}^t)]. \end{aligned} \tag{3}$$

Here, the conduction coefficients are updated at every step,

$$\begin{aligned} c_{i+\frac{1}{2},j} &= c(|(\nabla v)_{i+\frac{1}{2},j}^t|) \sim c(|v_{i+1,j}^t - v_{i,j}^t|), \\ c_{i-\frac{1}{2},j} &= c(|(\nabla v)_{i-\frac{1}{2},j}^t|) \sim c(|v_{i-1,j}^t - v_{i,j}^t|), \\ c_{i,j+\frac{1}{2}} &= c(|(\nabla v)_{i,j+\frac{1}{2}}^t|) \sim c(|v_{i,j+1}^t - v_{i,j}^t|), \\ c_{i,j-\frac{1}{2}} &= c(|(\nabla v)_{i,j-\frac{1}{2}}^t|) \sim c(|v_{i,j-1}^t - v_{i,j}^t|). \end{aligned}$$

The choice of the analytic form of the nonlinear function $c(\cdot)$ leads to different behavior for (2). Following the analysis on robust anisotropic diffusion given by Black et al, [2], this work considers Tukey's biweight robust norm for $c(\cdot)$,

$$c(s;\sigma) = \begin{cases} .5\left[1 - \left(\frac{s}{\sigma}\right)^2\right]^2, & |s| \le \sigma, \\ 0, & \text{else.} \end{cases} \tag{4}$$

An analysis of the noise-reducing effects of nonlinear anisotropic diffusion are presented in [3].

Results

The iteration (2) depends sensitively on two factors:

- the parameter σ in $c(s;\sigma)$;
- the number of iterations chosen.

These factors are dependent on each other and may lead to different results if chosen inappropriately. The numerical scheme (3) requires the selection of an additional rate parameter λ which is related to the CFL condition of the scheme. In this work, $\lambda = .0312$.

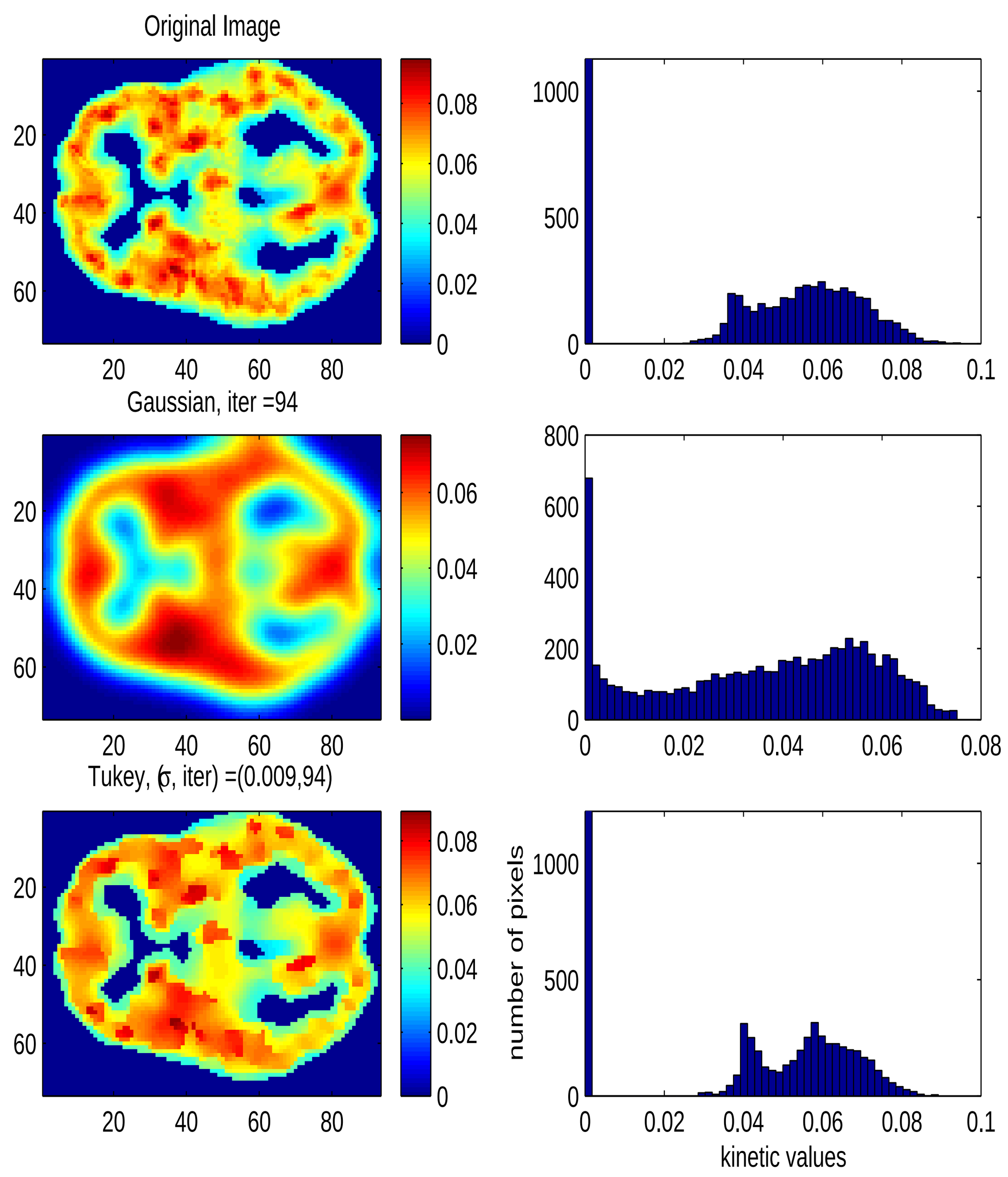
The choice of σ is motivated by considering a robust statistical measure, not affected by values of extreme observations.

The *Median Absolute Deviation* (MAD) is robust in this sense, [2]

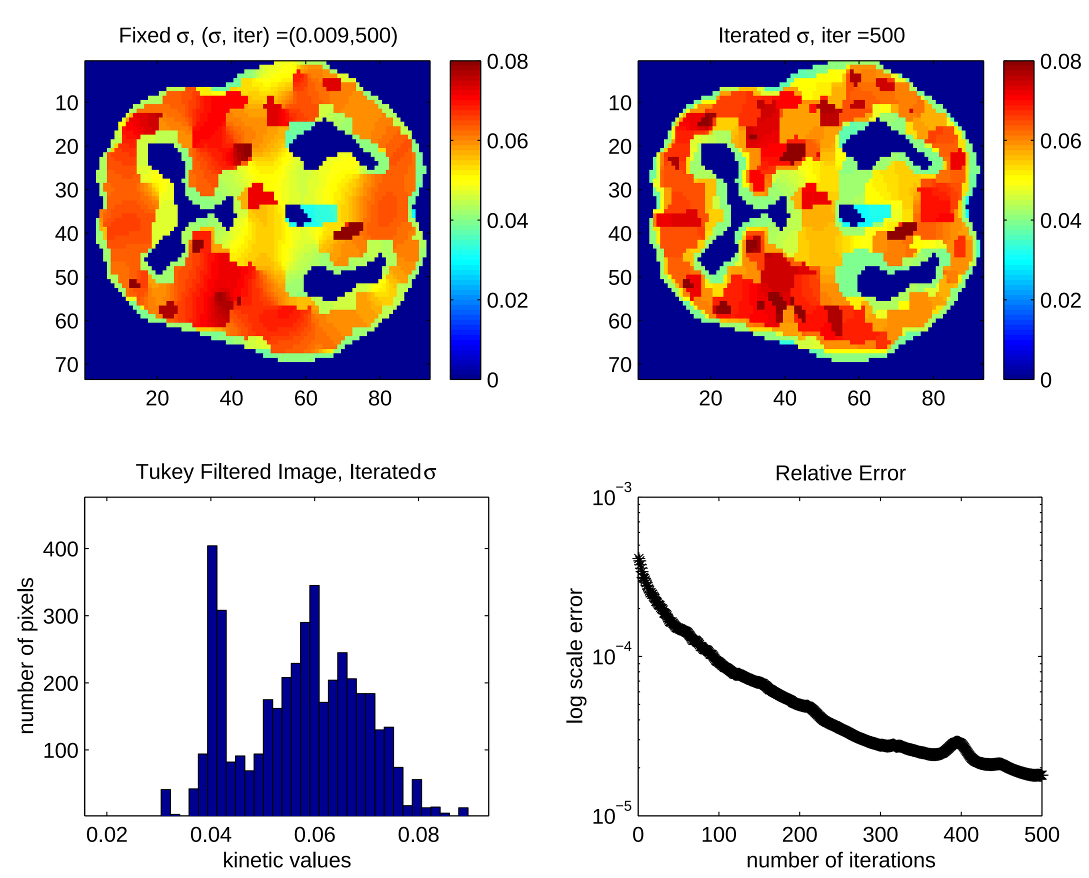
$$MAD = median_{\mathbf{v}}||\nabla \mathbf{v} - median_{\mathbf{v}}(||\nabla \mathbf{v}||)||.$$

The iteration is stopped if $\frac{||\mathbf{v^{t+1}} - \mathbf{v^t}||}{||\mathbf{v^{t+1}}||} \le tol$, for a given tolerance. The images below were generated as the solution to the inverse problem given in (1) for the dynamics of Fluoro-Deoxy-Glucose tracer uptake in the normal brain, see [4] for more details on this generation. The parameter vector $k_j = [k_1, k_2, k_3, k_4]_j$ describes respectively, forward transport from plasma into tissue, back-transport from tissue into blood, phosphoryllation and de-phosphorylation of FDG in tissue. In addition, another parameter of interest is $K = \frac{k_1 k_3}{k_2 + k_3}$, a proportionality constant for the rate of glucose uptake. The results below were obtained by applying anisotropic filtering to K and k_4 images.

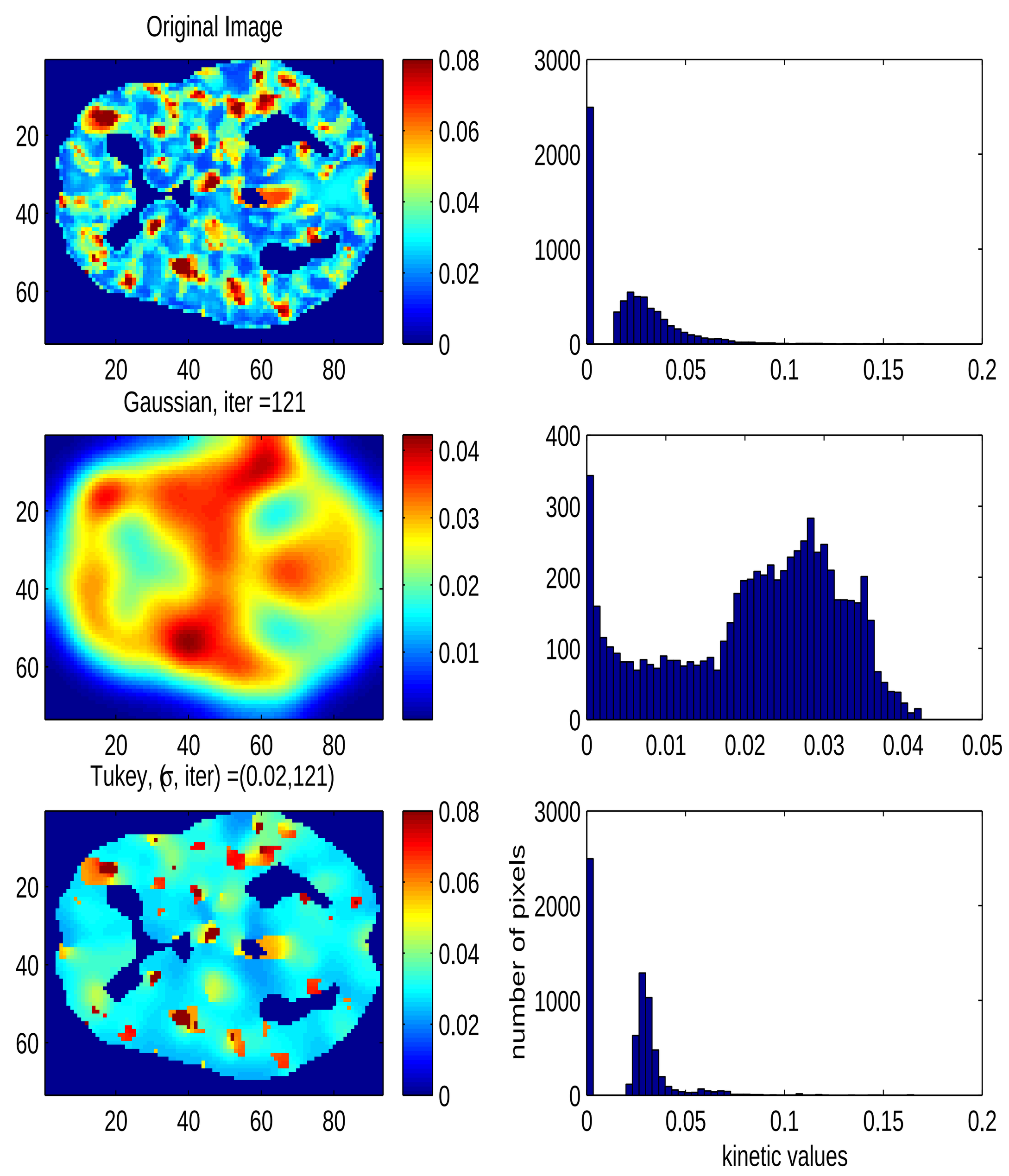
Gaussian filtering is compared with anisotropic filtering. *Results for* K follow.

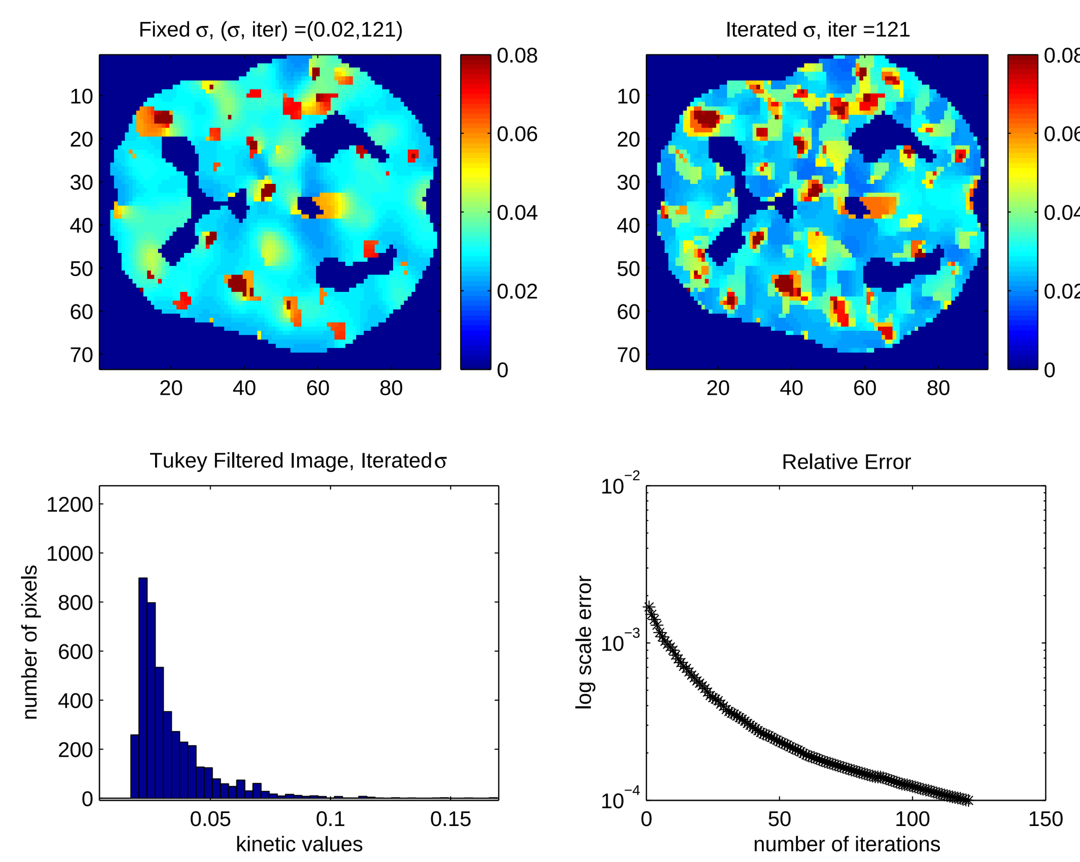


In addition to using a fixed parameter σ proportional to MAD througout the iteration (3), we also consider an iterative update of σ based on the median absolute deviation computed at each step based on the updated image.



Similar results were obtained for the de-phosphorylation parameter k_4. Note, this parameter does not follow a Gaussian distribution as K does. *Results for* k_4 follow.





Conclusions

This study uses anisotropic diffusion to filter global kinetic images derived from Positron Emission Tomography data.

- Using a Gaussian filter for kinetic PET data leads to blurring, and hence to a possible misclassification of regional function.
- Anisotropic filtering is a smooth, edge enhancing filter and thus better suited for maintaining region specific functional information.
- The iterative update of σ, proportional to the median absolute deviation, maintains sharper edges than the iterative update using a fixed σ value.

References

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